Seminar: Sheaf Theory

Fine, soft and flabby sheaves

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Contents

1 Sheaf Cohomology and Acyclic resolutions **3**

2 Flabby Sheaves and Resolutions **5**

3 Soft and Fine Sheaves **8**

3.1 Topological Preliminaries 8

3.2 Soft Sheaves 9

3.3 Fine Sheaves 12

4 Conclusion **13**

1 Sheaf Cohomology and Acyclic resolutions

Let R be a ring and $(X, \mathcal{O}_X)$ a ringed space over R.

Definition 1.1. For an $\mathcal{O}_X$-Module $\mathcal{F}$, we defined the *cohomology* of $\mathcal{F}$ as the right derivation of the left exact functor $\Gamma(X, -) : \mathcal{O}_X\text{-Mod} \to R\text{-Mod}, \mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$: For $n \in \mathbb{N}_0$

$$H^n(X, \mathcal{F}) = R^n\Gamma(X, \mathcal{F}).$$

Remark. This also defines the cohomolgy of a sheaf $\mathcal{F}$ of abelian groups on X, because $\mathcal{F}$ can be regarded as a $\underline{\mathbb{Z}}$-module, $\underline{\mathbb{Z}}$ being the constant $\mathbb{Z}$-valued sheaf on X.

For the theoretic construction, injective resolutions were used, but we learned that we can use a greater class of objects to resolve $\mathcal{F}$ for easier computation.

Definition 1.2. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $\mathcal{A}$ with enough injectives, $F : \mathcal{A} \to \mathcal{B}$ a covariant left exact functor. We call $I \in \mathcal{A}$ *F-acyclic* iff for all $n > 0$ we have $R^nF(I) = 0$.

Let A be any object in $\mathcal{A}$. An *acyclic resolution* of A is an exact sequence $0 \to A \to I^0 \to I^1 \to \dots$, where for all $n \geq 0$, I^n is F-acyclic.

Theorem 1.3. *Let $0 \to A \to I^\bullet$ be an F-acyclic resolutions of A, then there is a natural isomorphism $R^nF(A) \cong H^n(F(\mathcal{I}^\bullet))$.*

We applied this to the section functor Γ and defined (Γ-)acyclic $\mathcal{O}_X$-modules. In this talk, we will look at different examples of acyclic sheaves and $\mathcal{O}_X$-modules.

A useful criterion to find classes of acyclic objects is the following:

Lemma 1.4. *Let $\mathcal{A}, \mathcal{B}, F : \mathcal{A} \to \mathcal{B}$ as above. Let $\mathcal{I} \subset \operatorname{Ob} \mathcal{A}$ be a subset such that the following conditions are satisfied:*

(i) If I is an injective object of $\mathcal{A}$, then $I \in \mathcal{I}$.

(ii) Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence in $\mathcal{A}$ with $X', X \in \mathcal{I}$. Then X'' is in $\mathcal{I}$ and $F(X) \to F(X'')$ is surjective.

Then every object in $\mathcal{I}$ is F-acyclic.

Proof. Let $A \in \mathcal{I}$. Since $\mathcal{A}$ has enough injectives, we get an injective resolution

$$0 \to A \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \dots$$

By (i), I^0 is in $\mathcal{I}$ and so we can apply (ii) to the short exact sequence

$$0 \to A \to I^0 \xrightarrow{d^0} \operatorname{im}(d^0) \to 0.$$

This shows that $\operatorname{im}(d^0) \in \mathcal{I}$ and that

$$0 \to F(A) \to F(I^0) \xrightarrow{F(d^0)} F(\operatorname{im}(d^0)) \to 0$$

is exact, i. e. $F(\operatorname{im}(d^0)) = \operatorname{im}(F(d^0))$.

This was the base case for our induction. Now let $n > 0$ and assume that $\operatorname{im}(d^{n-1}) \in \mathcal{I}$ and $F(\operatorname{im}(d^{n-1})) = \operatorname{im}(F(d^{n-1}))$.

$I^n \in \mathcal{I}$ because of (i), so we can apply (ii) to the s.e.s.

$$0 \to \operatorname{im}(d^{n-1}) = \ker(d^n) \to I^n \xrightarrow{d^n} \operatorname{im}(d^n) \to 0.$$

This means that also $\operatorname{im}(d^n) \in \mathcal{I}$ and gives us exactness of the sequence

$$0 \to F(\operatorname{im}(d^{n-1})) \to F(I^n) \xrightarrow{F(d^n)} F(\operatorname{im}(d^n)) \to 0.$$

Hereby we can conclude that

$$\operatorname{im}(F(d^{n-1})) = F(\operatorname{im}(d^{n-1})) = \ker(F(d^n))$$

and hence $R^nF(A) = \dfrac{\ker(F(d^n))}{\operatorname{im}(F(d^{n-1}))} = 0.$ □

2 Flabby Sheaves and Resolutions

In this chapter, let X be a topological space.

Definition 2.1. We call a sheaf $\mathcal{F}$ on X *flabby* (sometimes also by the french word *flasque*), if for every open subset $U \subseteq X$ the restriction map $\rho_{XU} : \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

Remark. Let $\mathcal{F}$ be a flabby sheaf on X and $U \subseteq X$ open. Then $\mathcal{F}|_U$ is a flabby sheaf on U: If $V \subseteq U$ is open, then it is also open in X. Therefore the composition of restrictions $\mathcal{F}(X) \to \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective and hence $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

The following example shows that flabby really means "functions can be extended".

Example 2.2. The sheaf of locally constant $\mathbb{Z}$-valued functions $\underline{\mathbb{Z}}$ on X is flabby if and only if X is *irreducible* (i. e. if any two nonempty open sets have nonempty intersection, also called *hyperconnected*), because in this case locally constant functions on a nonempty open subset $U \subseteq X$ are already constant and $\underline{\mathbb{Z}}(U) \cong \mathbb{Z}$. An example for an irreducible space is $\mathbb{C}$ with the Zariski topology, which is just the cofinite topology, where the closed sets are exactly the finite ones. On the other hand, $\underline{\mathbb{Z}}$ on $\mathbb{C}$ with the standard topology is not flabby, because we can find two nonempty disjoint subsets, e. g. $U = B_1(-2)$ and $V = B_1(2)$. Then the function $f \in \underline{\mathbb{Z}}(U \cup V)$ with $f|_U = 0$ and $f|_V = 1$ can not be extended to a locally constant function on $\mathbb{C}$.

We will now see an easy, canonical, and *functorial* way to construct flabby resolutions for any sheaf or $\mathcal{O}_X$-module. Later, we will see that this is in fact an acyclic resolution, allowing us to compute cohomology.

Definition 2.3 (Godement-Functor)**.** Let $\mathcal{F}$ be a sheaf on X. We define the sheaf $\mathcal{F}_{\text{God}}$ as follows: For every open subset $U \subseteq X$, we set

$$\mathcal{F}_{\text{God}}(U) = \prod_{x \in U} \mathcal{F}_x$$

with the obvious restriction maps. Note that in this case, they are actually restrictions, because an element $s \in F_{\text{God}}(U)$ can be explicitly regarded as a function $U \to \coprod_{x \in U} \mathcal{F}_x$, which we can restrict to V. With this in mind, it is easy to see that $\mathcal{F}_{\text{God}}$ is indeed a sheaf and flabby.

Furthermore, there is an injective morphism of sheaves $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}_{\text{God}}$, that for every $U \subseteq X$ open maps $s \in \mathcal{F}(U)$ to $(s_x)_{x \in U} \in \mathcal{F}_{\text{God}}(U)$.

Given a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves, we define $\phi_{\text{God}} : \mathcal{F}_{\text{God}} \to \mathcal{G}_{\text{God}}$ with $\phi_{\text{God}\,U} = \prod_{x \in U} \phi_x$. This is a morphism of sheaves.

Hereby we obtain a functor $-_{\text{God}}$ from the category of sheaves to the full subcategory of flabby sheaves. The morphism $\iota_{\mathcal{F}}$ is functorial in $\mathcal{F}$.

If $(X, \mathcal{O}_X)$ is a ringed space and the sheaf $\mathcal{F}$ that we started with is an $\mathcal{O}_X$-module, $\mathcal{F}_{\text{God}}$ is as well: Addition is defined pointwise in the stalks, and scalar multiplication of $a \in \mathcal{O}_X(U)$ with $(s_x)_{x\in U}$ is defined as $(a_x s_x)_{x\in U} \in \mathcal{F}_{\text{God}}(U)$. Hence we get a functor $-_{\text{God}}$ from the category of $\mathcal{O}_X$-modules to the full subcategory of flabby $\mathcal{O}_X$-modules. Here, $\iota_{\mathcal{F}}$ is a functorial homomorphism of $\mathcal{O}_X$-modules.

Definition 2.4 (Godement-Resolution)**.** Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Using the above construction, we set $\mathcal{F}^0 := \mathcal{F}_{\text{God}}$ and get a functorial exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \xrightarrow{\iota_{\mathcal{F}} =: \delta_{-1}} \mathcal{F}^0 \to \operatorname{coker}(\iota_{\mathcal{F}}) \to 0$$

Now $(coker(\iota_{\mathcal{F}}))_{\text{God}} =: \mathcal{F}^1$ is a good candidate to continue our resolution, as we already know it is flabby. Inductively, we define for $n > 0$:

- $\mathcal{F}^{n+1} := (\operatorname{coker}(\delta_{n-1}))_{\text{God}}$
- The morphism $\delta_n : \mathcal{F}^n \to \mathcal{F}^{n+1}$ is defined as the composition $\mathcal{F}^n \to \operatorname{coker}(\delta_{n-1}) \to (\operatorname{coker}(\delta_{n-1}))_{\text{God}}$. Because the second arrow is injective, we get the exactness of the resulting flabby resolution

$$0 \to \mathcal{F} \xrightarrow{\iota_{\mathcal{F}}} \mathcal{F}^0 \xrightarrow{\delta_0} \mathcal{F}^1 \xrightarrow{\delta_1} \mathcal{F}^2 \xrightarrow{\delta_2} \dots$$

Remark. In general, derived functors depend on the choice of injective resolution and thus are only well-defined up to natural isomorphism. For the special case of sheaf cohomology though, we now have a canonical and functorial way to compute it, so every $H^n(X, -)$ is a well-defined functor.

Example 2.5. If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{I}$ an injective $\mathcal{O}_X$-module, then $\mathcal{I}$ is flabby.

Proof. By Definition 2.3, there is a flabby $\mathcal{O}_X$-module $\mathcal{I}_{\text{God}}$ and an injective homomorphism $\mathcal{I} \to \mathcal{I}_{\text{God}}$. Because $\mathcal{I}$ is injective, it is thus a direct summand of $\mathcal{I}_{\text{God}}$ and hence flabby itself. (This works because a *finite* direct sum is a direct product, so for every $U \subseteq X$ we can write $\mathcal{I}_{\text{God}}(U) = \mathcal{I}(U) \oplus \mathcal{I}'(U)$ without sheafifying). □

Lemma 2.6. *Let $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \to 0$ be a short exact sequence of sheaves with $\mathcal{F}', \mathcal{F}$ flabby. Then $\beta_X : \mathcal{F}'(X) \to \mathcal{F}''(X)$ is surjective and $\mathcal{F}''$ is flabby.*

Proof. We can consider $\mathcal{F}'$ as a subsheaf of $\mathcal{F}$, more specifically as $\ker \beta$. Now let $s \in \mathcal{F}''(X)$. We define the set

$$\mathcal{C} := \{(U, t) \mid U \subseteq X \text{ open}, t \in \mathcal{F}(U) \text{ s. t. } \beta_U(t|_U) = s|_U\},$$

i. e. we look at subsets of X where we can find a preimage for s. Our goal is to find some $t \in \mathcal{F}(X)$ such that $(X, t) \in \mathcal{C}$.

$\mathcal{C}$ is nonempty, because $(\emptyset, *) \in \mathcal{C}$. We can order $\mathcal{C}$ by $(U,t) \leq (U',t') \Leftrightarrow U \subseteq U'$ and $t'|_U = t$. Then it is inductively ordered by this relation, i. e. every chain has an upper bound given by the union of the sets and the glued-together sections. We apply *Zorn's Lemma* to get a maximal element of $\mathcal{C}$, say (V,t). We now proof that $V = X$.

Suppose there exists $x \in X \setminus V$. Then there is a neighborhood W of x and $t' \in \mathcal{F}(W)$ such that $(W,t') \in \mathcal{C}$ (because we know $\beta_x : \mathcal{F}_x \to \mathcal{F}''_x$ is surjective, that means β is surjective "near" x). Now, since both t and t' get mapped to s on $V \cap W$, we get $t|_{V\cap W} - t'|_{V\cap W} \in \ker \beta(V \cap W) = \mathcal{F}'(V \cap W)$. Since $\mathcal{F}'$ is flabby, we can extend this to some $t'' \in \mathcal{F}'(W) \subseteq \mathcal{F}(W)$. Then, t and $t' + t''$ agree on $V \cap W$ and can be glued together to a section in $\mathcal{F}(V \cup W)$ that extends t and gets mapped to $s|_{V\cup W}$ under β, contradicting the maximality of V.

Now to show that $\mathcal{F}''$ is flabby, we can use what we've just shown. For $U \subseteq X$ open, we get surjectivity of $\beta_U : \mathcal{F}'(U) \to \mathcal{F}''(U)$ by applying the same proof as above to $\mathcal{F}'|_U, \mathcal{F}|_U, \mathcal{F}''|_U$, since $\mathcal{F}'|_U$ and $\mathcal{F}|_U$ are obviously also flabby. Then it is obvious from the diagram that the right vertical arrow must also be surjective.

$$\begin{array}{ccc} \mathcal{F}(X) & \twoheadrightarrow & \mathcal{F}''(X) \\ \downarrow\!\!\!\downarrow & & \downarrow \\ \mathcal{F}(U) & \twoheadrightarrow & \mathcal{F}''(U) \end{array}$$

□

Corollary 2.7. *All flabby sheaves (of abelian groups) are* Γ*-acyclic.*

Proof. This follows from Lemma 1.4, Example 2.5 and Lemma 2.6. □

3 Soft and Fine Sheaves

3.1 Topological Preliminaries

Definition 3.1. A topological space X is called *paracompact* if every open covering of X has an open refinement that is locally finite.

Remark. Every closed subspace of a paracompact space is paracompact.

Definition 3.2. A topological space X is called *Hausdorff space* if for every two points $x, y \in X, x \neq y$ there exist open neighbourhoods $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$.

Theorem 3.3 (Tietze Extension Theorem)**.** *Let X be a topological space. Then the following properties are equivalent:*

(i) For any two closed subsets $A, B \subseteq X$ with $A \cap B = \emptyset$, there exists a continuous function $f : X \to [0,1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

(ii) For any two closed subsets $A, B \subseteq X$ with $A \cap B = \emptyset$, there exist open disjoint sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(iii) For every closed subset A and every neighborhood W of A there exists an open neighborhood U of A such that $A \subseteq U \subseteq \overline{U} \subseteq W$.

(iv) For every closed subspace $A \subseteq X$ and continuous map $f : A \to \mathbb{R}$ there exists a continuous function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}|_A = f$.

Definition 3.4. A topological space X that satisfies the above properties is called *normal.*

Lemma 3.5 (Shrinking Lemma)**.** *Let X be a normal topological space and let $(U_i)_{i\in I}$ be a locally finite open covering. Then there exists another open covering $(V_i)_{i\in I}$ such that for all $i \in I$, $V_i \subseteq \overline{V_i} \subseteq U_i$, i. e. we can shrink the original sets to get a locally finite covering by* closed *sets.*

Remark. This assumes the Axiom of Choice.

Proposition 3.6. *Every paracompact Hausdorff space is normal.*

Example 3.7. Differential manifolds over $\mathbb{R}, \mathbb{C}$ are paracompact Hausdorff.

3.2 Soft Sheaves

Definition 3.8. Let X be a topological space and $\mathcal{F}$ a sheaf on X. For a subspace $Z \subseteq X$, we define

$$\mathcal{F}(Z) := \left\{ (s_i, U_i)_{i\in I} \,\middle|\, U_i \subseteq X \text{ open with } Z \subseteq \bigcup_{i\in I} U_i, \right.$$
$$\left. s_i \in \mathcal{F}(U_i) \text{ s.t. } \forall\, i, i' \in I, z \in Z \cap U_i \cap U_{i'} : (s_i)_z = (s_{i'})_z \right\} /_\sim$$

with $(s_i, U_i)_{i\in I} \sim (t_j, V_j)_{j\in J} \Leftrightarrow \forall\, i \in I, j \in J, z \in Z \cap U_i \cap V_j : (s_i)_z = (t_j)_z$.

For every open neighborhood U of Z, we have a restriction map $\rho_Z^U : \mathcal{F}(U) \to \mathcal{F}(Z), s \mapsto s|_Z := [(U, s)]$

Remark. (i) The restriction of $\mathcal{F}$ to a subspace Z is usually defined as $\mathcal{F}(Z) = (\iota^{-1}\mathcal{F})(Z)$, using the inverse image sheaf of the inclusion $\iota : Z \to X$. Since we did not cover this in the seminar, I'm giving an explicit description instead.

(ii) If $Z \subseteq X$ is open, this gives us the usual $\mathcal{F}(Z)$.

(iii) For the case $Z = \{x\}$, this just gives us the stalk $\mathcal{F}_x$.

Definition 3.9. A sheaf $\mathcal{F}$ on a topological space X is called *soft* if for every closed $Z \subseteq X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(Z)$ is surjective.

Remark. Let X be a topological space, $Z \subseteq X$ closed and $\mathcal{F}$ a soft sheaf on X. Then $\mathcal{F}|_Z$ is a soft sheaf on Z: If $A \subseteq Z$ is closed, then it is also closed in X. Therefore the composition of restrictions $\mathcal{F}(X) \to \mathcal{F}(Z) \to \mathcal{F}(A)$ is surjective and hence $\mathcal{F}(Z) \to \mathcal{F}(A)$ is surjective.

Proposition 3.10. *Assume that X is a paracompact Hausdorff space, $Z \subseteq X$ closed and $\mathcal{F}$ a sheaf on X. Then the restriction maps ρ_Z^U induce an isomorphism*

$$\operatorname*{colim}_{Z\subseteq U} \mathcal{F}(U) = \{(U, s) \,|\, U \subseteq X \textit{ open with } Z \subseteq U, s \in \mathcal{F}(U)\}/_\sim \cong \mathcal{F}(Z)$$

where in the middle term $(U, s) \sim (V, t) \Leftrightarrow$ there exists $Z \subseteq W \subseteq U \cap V$ open such that $s|_W = t|_W$.

Proof. Uses some topological results. See [Wed16], Proposition 9.1. □

Example 3.11. (i) Let X be a paracompact Hausdorff space and let $\mathcal{C}_X$ be the sheaf of continuous $\mathbb{R}$-valued functions on X. Then $\mathcal{C}_X$ is soft. (It is however, in general, not flabby: Let $X = \mathbb{R}$ with the standard topology. The continuous function $s : (0, \infty) \to \mathbb{R}, x \mapsto \frac{1}{x}$ can not be extended to $\mathbb{R}$).

Proof. To show this, let $Z \subseteq X$ closed and $s \in \mathcal{C}_X(Z)$. Note that in general, $\mathcal{C}_X(Z) \neq \mathcal{C}_Z(Z)$, so we cannot directly apply Theorem 3.3 (iv)! Instead, we use Proposition 3.10 to get an open neighborhood $Z \subseteq U \subseteq X$ and $t \in \mathcal{C}_X(U)$ that extends s. We can then use Theorem 3.3 (iii) to find an open neighborhood $Z \subseteq V$ such that $\overline{V} \subseteq U$. Then Z and $X \setminus V$ are disjoint closed subsets of X and by Theorem 3.3 (i) we get a continuous function $f : X \to [0, 1]$ with $f|_Z = 1$ and $f_{X \setminus V} = 0$.

Thus $f|_U t$ also extends s (i. e. $(f|_U t)|_Z = t_Z = s_Z$) and can be extended to X by zero. □

(ii) It can be shown in a similar way that for X a real $\mathcal{C}^\alpha$-manifold, $\alpha \in \mathbb{N} \cup \infty$, the sheaf of real valued $\mathcal{C}^\alpha$-functions on X is soft. The proof relies on the fact that every point in X has a compact neighborhood that corresponds to a compact set $K \in \mathbb{R}^m$ via a chart. We can then use the fact that for all $U \in \mathbb{R}^m$ open and $K \in U$ compact, there exists a C^∞-function $\phi : \mathbb{R}^m \to [0, 1]$ with $\phi|_K = 1$, $\operatorname{supp} \phi \subseteq U$ to extend functions by zero.

(iii) We cannot use the same trick for holomorphic functions on complex manifolds: The set of zeros of a holomorphic function is discrete, so if we extend a holomorphic function by zero on the complement of a compact set, the result can not be holomorphic.

(iv) Now, for example the sheaf of holomorphic functions $\mathcal{O}$ on $\mathbb{C}$ is not soft, as we can easily see: Let $Z \subseteq \mathbb{C}$ closed such that $Z \subseteq \mathbb{C}^- := \mathbb{C} \setminus \{x \in \mathbb{R} \,|\, x \leq 0\}$, and take the element $s \in \mathcal{O}(Z)$ represented by the complex logarithm ln. Due to the identity theorem, there are no other representatives and there is no globally holomorphic function on $\mathcal{C}$ that extends ln. (Another example: $z \mapsto \sum_{n=0}^{\infty} z^n$ on $Z \subseteq \{z \in \mathcal{C} | z| < \frac{1}{2}\}$).

(v) Constant sheaves are in general not soft. Take for example $\underline{\mathbb{Z}}$ on $X = \mathbb{C}$ with the Zariski topology. We saw before that this was flabby. But it is not soft, as the section $s \in \underline{\mathbb{Z}}(\{0, 1\})$ with $s_0 = 0, s_1 = 1$ can not be extended to the whole space. (Remember that we must use Definition 3.8 here, not Proposition 3.10, to find that s).

Example 3.12. Let X be a paracompact Hausdorff space. Then every flabby sheaf $\mathcal{F}$ on X is soft.

Proof. Let $Z \subseteq X$ be closed, and $s \in \mathcal{F}(Z)$. By Proposition 3.10 there exists an open neighborhood U of Z and $t \in \mathcal{F}(U)$ extending s. Since $\mathcal{F}$ is flabby, we can extend t to X. □

Proposition 3.13. *Let $(X, \mathcal{O}_X)$ be a ringed space with X paracompact Hausdorff and O_X soft. Then every $\mathcal{O}_X$-module $\mathcal{F}$ is soft.*

Proof. Let $Z \subseteq X$ be closed and $s \in \mathcal{F}(Z)$. By Proposition 3.10 there exists an open neighborhood U of Z and $t \in \mathcal{F}(U)$ extending s. X is normal, and so we can use

Theorem 3.3 (iii) to find an open neighborhood $Z \subseteq V$ such that $\overline{V} \subseteq U$. Then Z and ∂V are disjoint closed subsets of X. Because $\mathcal{O}_X$ is soft, $\mathcal{O}_X(X) \to \mathcal{O}_X(Z \cup \partial V)$ is surjective and we can find $u \in \mathcal{O}_X(X)$ such that for all $x \in Z$ we have $u_x = 1$ and for $x \in \partial V$ we have $u_x = 0$. Thus $u|_{\overline{V}} t|_{\overline{V}} \in \mathcal{F}(\overline{V})$ extends s and can be extended to an element of $\mathcal{F}(X)$ by zero. □

Theorem 3.14. *Let $(X, \mathcal{O}_X)$ be a ringed space with X paracompact Hausdorff. Then every soft $\mathcal{O}_X$-module is Γ-acyclic.*

Proof. Again, we want to use Lemma 1.4 to proof this. Examples 2.5 and 3.12 show that every injective $\mathcal{O}_X$-module is flabby and thus soft, so it remains to show that the second condition of the Lemma is fulfilled. So let $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules with $\mathcal{F}', \mathcal{F}$ soft. As in the proof of Lemma 2.6, we consider $\mathcal{F}'$ as a subsheaf of $\mathcal{F}$, more specifically as $\ker \beta$. To show surjectivity of $\beta_X : \mathcal{F}(X) \to \mathcal{F}''(X)$, let $s \in \mathcal{F}''(X)$. We know that for all $x \in X, \beta_x : \mathcal{F}_x \to \mathcal{F}''_x$ is surjective, that means β is surjective "near" any x. So we can find an open covering $(U_i)_{i \in I}$ of X such that for all $i \in I$, there is a section $s_i \in \mathcal{F}(U_i)$ with $\beta_{U_i}(s_i) = s|_{U_i}$. Because X is paracompact, we can assume $(U_i)_{i \in I}$ is locally finite. We use the Shrinking Lemma 3.5 to find another open covering $(V_i)_{i \in I}$ such that for all $i \in I$ the closure $S_i := \overline{V_i}$ is contained in U_i. For $J \subseteq I$ set $S_J := \bigcup_{i \in J} S_i$. Note that this is closed, because $(S_i)_{i \in I}$ is locally finite.

We define the set

$$\mathcal{C} := \{(J, t) \mid J \subseteq I, t \in \mathcal{F}(S_J) \text{ s. t. } \beta_{S_J}(t|_{S_J}) = s|_{S_J}\},$$

$\mathcal{C}$ is nonempty, because $(\emptyset, *) \in \mathcal{C}$. We can order $\mathcal{C}$ by $(J, t) \leq (J', t') \Leftrightarrow J \subseteq J'$ and $t'|_{S_J} = t$. Then it is inductively ordered by this relation, i. e. every chain has an upper bound given by the union of the sets and the glued-together sections. We apply *Zorn's Lemma* to get a maximal element of $\mathcal{C}$, say (J, t). We now proof that $J = I$, then t is a preimage of s on $S_I = X$.

Suppose there exists $i \in I \setminus J$. Then we have $s_i|_{S_i} \in \mathcal{F}(S_i)$ that gets mapped to $s|_{S_i}$ under β. Now, since both t and s_i get mapped to S on $S_J \cap S_i$, we get $t|_{S_J \cap S_i} - s_i|_{S_J \cap S_i} \in \ker \beta(S_J \cap S_i) = \mathcal{F}'(S_J \cap S_i)$. Since $\mathcal{F}'$ is soft, we can extend this to some $t' \in \mathcal{F}'(S_i) \subseteq \mathcal{F}(S_i)$. Then, t and t' agree on $S_J \cap S_i$ and can be glued together to a section in $\mathcal{F}(S_J \cap S_i)$ that extends t and gets mapped to $s|_{S_J \cap S_i}$ under β, contradicting the maximality of J.

Now to show that $\mathcal{F}''$ is soft, let $Z \subseteq X$ be closed. We can apply what we've just shown to $\mathcal{F}'|_Z, \mathcal{F}|Z$ and $\mathcal{F}''|_Z$, because the first two are again soft. Then it is obvious from the diagram that the right vertical arrow must be surjective.

$$\begin{array}{ccc} \mathcal{F}(X) & \twoheadrightarrow & \mathcal{F}''(X) \\ \downarrow\!\!\downarrow & & \downarrow \\ \mathcal{F}(Z) & \twoheadrightarrow & \mathcal{F}''(Z) \end{array}$$

□

Theorem 3.15. *Let X be a paracompact Hausdorff Space and $\mathcal{F}$ a sheaf of abelian groups on X. Then $\mathcal{F}$ is soft iff for every closed subspace $Z \subseteq X$, every $s \in \mathcal{F}(Z)$ and every open covering $(U_i)_{i \in I}$ of Z in X, for all $i \in I$ there exists $s_i \in \mathcal{F}(X)$ with* $\operatorname{supp}(s_i) \subseteq U_i$ *and for all $x \in Z$ we have $s_x = \sum_{i \in I} (s_i)_x$.*

Suppose that $\mathcal{F}$ is a sheaf of rings. Then it is soft iff the condition holds for $Z = X$ and $s = 1$, i. e. we have partitions of unity.

Proof. The direction $\Rightarrow$ can be seen in [Wed16], Proposition 9.9. The rest is omitted. □

3.3 Fine Sheaves

Definition 3.16 (Sheaf of homomorphisms of $\mathcal{O}_X$-modules)**.** Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. The presheaf

$$U \mapsto \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

with the obvious restriction maps is a sheaf. The right hand side is a $\mathcal{O}_X(U)$-module. Therefore this sheaf has the structure of an $\mathcal{O}_X$-module and we will denote it by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

Definition 3.17. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. $\mathcal{F}$ is said to be *fine* if $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ is soft.

Proposition 3.18. *Let $(X, \mathcal{O}_X)$ be a ringed space with X paracompact Hausdorff and $\mathcal{O}_X$ soft. Then every $\mathcal{O}_X$-module $\mathcal{F}$ is fine.*

Proof. $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ is an $\mathcal{O}_X$-module, and as such it is soft by Proposition 3.13. □

Remark. If we view $\mathcal{O}_X$ as a module over itself, the notions soft and fine are equivalent.

Remark. Using Theorem 3.15 and Proposition 3.18, we see that a sheaf on a paracompact Hausdorff space that is fine by [Voi02], Def. 4.35 is also fine by our Definition.

Proposition 3.19. *Let $(X, \mathcal{O}_X)$ be a ringed space with X paracompact Hausdorff and let $\mathcal{F}$ be a fine $\mathcal{O}_X$-module. Then it is soft, and by consequence Γ-acyclic.*

Proof. We can view $\mathcal{F}$ as a module over the soft sheaf of rings $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$, using the evaluation as scalar multiplication. Then it is soft by Proposition 3.13. □

4 Conclusion

For the general case of X any topological space and $\mathcal{F}$ an abelian sheaf on X, we saw the following implications:

$$\mathcal{F} \text{ injective} \implies \mathcal{F} \text{ flabby} \implies \mathcal{F} \text{ acyclic}$$

Because we have functorial flabby resolutions using the Godement-construction, we can view sheaf cohomology as a functor.

We then added some more structure, namely looking at ringed spaces $(X, \mathcal{O}_X)$ with X paracompact Hausdorff. Then we additionally have for an $\mathcal{O}_X$-module $\mathcal{F}$:

$$\begin{array}{ccccc} \mathcal{F} \text{ injective} \implies \mathcal{F} \text{ flabby} & & & & \\ & \searrow\!\!\!\searrow & & & \\ & & \mathcal{F} \text{ soft} & \implies & \mathcal{F} \text{ acyclic} \\ & \nearrow\!\!\!\nearrow & & & \\ \mathcal{F} \text{ fine} & & & & \end{array}$$

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